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Spatially localized, temporally quasiperiodic, discrete nonlinear excitations

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In contrast to the commonly discussed discrete breather, which is a spatially localized, time-periodic solution, we present an exact solution of a discrete nonlinear Schrödinger breather which is a spatially localized, temporally quasiperiodic nonlinear coherent excitation. This breather is a multiple-soliton solution in the sense of the inverse scattering transform. A discrete breather of multiple frequencies is conceptually important in studies of nonlinear lattice systems. We point out that, for this breather, the incommensurability of its frequencies is a discrete lattice effect and these frequencies become commensurate in the continuum limit. To understand the dynamical properties of the breather, we also discuss its stability and its behavior in the presence of an external potential. Finally, we indicate how to obtain an exact N-soliton breather as a discrete generalization of the continuum multiple-soliton solution.

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Historically, searching for breather solutions in the continuum and discrete ϕ^4 field equations has deepened our understanding of discrete lattice effects and their intricate interplay with integrability vs nonintegrability [1]. Since Sievers and Takeno put forth the idea of intrinsic localized states [2], the concept of breathers is also found to play a central role in the general theory of nonlinear lattice systems [3], particularly regarding self-focusing and collapse phenomena. Recently, a mathematical proof of the existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators has been given [4]. We note that these works mostly concentrate on the breather which is a spatially localized, and time-periodic nonlinear coherent excitation. Since nonlinearity precludes the separation of Fourier modes in time, it is natural to question the existence of temporally quasiperiodic or nonperiodic, spatially localized, nonlinear coherent excitations. This concept is important for the understanding of spatially localized nonlinear excitations in certain classical and quantum systems in that these excitations may intrinsically contain multiple internal frequencies, and are not reducible to any periodic structures in time. In this work, we will present an exact solution of a discrete breather which possesses multiple frequencies and is a bona fide soliton solution in the sense of the inverse scattering transform (IST). We will discuss the dynamical properties of this solution and find that it is actually a discrete generalization of the continuum multiple-soliton solution studied in Ref. [5]. As will be discussed below, these breathers with incommensurate internal breathing frequencies exist only in the discrete lattice. They approach time-periodic solutions in the continuum limit.

The system which we study is a discrete nonlinear Schrödinger equation (NLS). As is well known, discrete NLS equations are prototypical nonlinear lattice systems in various fields, such as condensed matter physics, molecular biology, and fiber optics [6]. The governing equation of our system is the Ablowitz-Ladik discretization of NLS [7]:

$$i\dot{\psi}_n=-(\psi_{n+1}+\psi_{n-1})-\mu(\psi_{n+1}+\psi_{n-1})|\psi_n|^2, \quad (1)$$

where $-2\psi_n$ in the finite difference Laplacian has been removed by a trivial gauge transformation. Due to the scaling property of Eq. (1), μ will be set to unity below. Since the initial condition, $\sinh\beta\,\mathrm{sech}[\beta(n-x_0)]$, gives rise to an envelope soliton, and the initial condition, $\sinh(2\beta)\mathrm{sech}[\beta(n-x_0)]$, evolves in time with its envelope executing a periodic motion [8], a candidate for the initial condition which will develop into a two frequency breathing structure, therefore, is

$$\psi_n^b(t=0)=\sinh(3\beta)\mathrm{sech}[\beta(n-x_0)]. \quad (2)$$

In order to confirm that this is indeed the case, and to obtain a breather solution of multiple frequencies, we will invoke the Hirota method [9]. The bilinear form for Eq. (1) is

$$(iD_t+2\cosh D_n)G_n\cdot F_n=0, \quad (3)$$

$$(\cosh D_n-1)F_n\cdot F_n=G_nG_n^*, \quad (4)$$

where $G_n/F_n=\psi_n$ with F_n being real, $2\cosh D_n(G_n\cdot F_n)=G_{n+1}F_{n-1}+G_{n-1}F_{n+1}$, and $D_t(G_n\cdot F_n)=[(\partial_t-\partial_{t'})G_n(t)F_n(t')]|_{t=t'}$. Using the asymptotic expansions, $F_n=\Sigma_{m=0}^{\infty}F_{2m,n}\epsilon^{2m}$, $G_n=\Sigma_{m=0}^{\infty}G_{2m+1,n}\epsilon^{2m+1}$, where $F_{0,n}=1$, we can derive a set of iterative equations involving $F_{2m,n}$ and $G_{2m+1,n}$ from Eqs. (3) and (4). The asymptotic expansions can be solved successively by starting a $G_{1,n}$ to solve for $F_{2,n}$, and then for $G_{3,n}$, and so forth. For the following choice of $G_{1,n}$,

$$G_{1,n}=a\exp(2it\cosh\beta)+b\exp(2it\cosh3\beta)$$
$$+c\exp(2it\cosh5\beta), \quad (5)$$

via lengthy algebraic manipulations, we can prove that the asymptotic expansions truncate at $O(\epsilon^7)$. In order for the wave function ψ_n to have the specific initial condition (2), the following values of a, b, and c are chosen:

$$a=2\sinh(3\beta)\exp(-\beta x_0), \quad (6)$$

$$b=4\sinh(3\beta)\sinh(4\beta)\cosh\beta\exp(-3\beta x_0)/\sinh\beta, \quad (7)$$

$$c = 4\sinh(3\beta)\sinh(5\beta)\cosh(2\beta)\exp(-5\beta x_0)/\sinh\beta, \quad (8)$$

with $x_0 \in (-\infty, +\infty)$ and $\beta \in (0, +\infty)$, which leads to the complicated expression for ψ_n:

$$\psi_n^b(t) = \sinh(3\beta)\exp(2it\cosh\beta)\frac{g_n(t)}{f_n(t)}, \quad (9)$$

where

$$g_n(t) = 2\cosh(8\beta\tilde{n}) + A_6\cosh(6\beta\tilde{n})\exp(i\Omega_{31}t) + [A_4\exp(i\Omega_{51}t) + B_4\exp(i\Omega_{31}t)]\cosh(4\beta\tilde{n}) + [A_2 + B_2\exp(i\Omega_{51}t)]\cosh(2\beta\tilde{n}) + C_1\exp(-i\Omega_{53}t) + C_2\exp(i\Omega_{53}t) + C_3\exp[i(\Omega_{53}+\Omega_{31})t], \quad (10)$$

$$f_n(t) = \cosh(9\beta\tilde{n}) + A_7\cosh(7\beta\tilde{n}) + A_5\cos(\Omega_{31}t)\cosh(5\beta\tilde{n}) + [A_3 + B_3\cos(\Omega_{51}t)]\cosh(3\beta\tilde{n}) + [A_1 + B_1\cos(\Omega_{53}t)]\cosh(\beta\tilde{n}), \quad (11)$$

where $\tilde{n} = n - x_0$, and

$$\Omega_{31} = 2(\cosh 3\beta - \cosh\beta), \quad (12)$$

$$\Omega_{51} = 2(\cosh 5\beta - \cosh\beta), \quad (13)$$

$$\Omega_{53} = 2(\cosh 5\beta - \cosh 3\beta); \quad (14)$$

$$A_1 = 4A_0\cosh^2 2\beta,$$

$$B_1 = 2A_0B_0,$$

$$A_2 = 32\cosh^2\beta\cosh^2 2\beta,$$

$$B_2 = 16B_0\cosh^2\beta\cosh 2\beta,$$

$$A_3 = 64\cosh^4\beta\cosh^2 2\beta,$$

$$A_4 = B_3 = 4B_0\cosh 2\beta,$$

$$A_5 = B_4 = 4A_0\cosh 2\beta,$$

$$A_6 = 16\cosh^2\beta\cosh 2\beta,$$

$$A_7 = A_0,$$

$$C_1 = B_0,$$

$$C_2 = A_0B_0,$$

$$C_3 = 4B_0\cosh^2 2\beta,$$

with $A_0 = (1 + 2\cosh 2\beta)^2$ and $B_0 = (1 + 2\cosh 2\beta + 2\cosh 4\beta)$.

It can be verified that, for $t = 0$, the initial condition for ψ_n^b [Eq. (9)] is indeed Eq. (2). From the perspective of IST, this solution can be viewed as a breather comprising three solitonic components with their poles at $\exp(-\beta)$, $\exp(-3\beta)$, and $\exp(-5\beta)$, respectively. Obviously, this breather evolves with three fundamental frequencies, the carrier-wave frequency $\Omega_c = 2\cosh\beta$ and the shape mode fundamentals Ω_{31} and Ω_{51}. Generally, these two shape mode frequencies are incommensurate, therefore, the envelope of the breather does not have a temporally periodic evolution. Since we are dealing with a nonlinear system, this excitation cannot be decomposed into three periodic components. This is an example of a spatially localized, temporally quasiperiodic, nonlinear breathing structure. In Fig. 1, such a breather is shown, in which the quasiperiodic characteristics can be clearly observed. We point out that these frequencies become commensurate in the continuum limit since $(\Omega_c - 2):\Omega_{31}:\Omega_{51} = 1:8:24$ in the limit of $\beta \rightarrow 0$. [Here $\Omega_c - 2$ is used in order to take into account the phase $\exp(2it)$ restored by the gauge transformation in taking the continuum limit.] Hence the evolution of the continuum breather is always periodic in time. The resonance situation of the system (1), when these frequencies become commensurate, deserves further consideration.

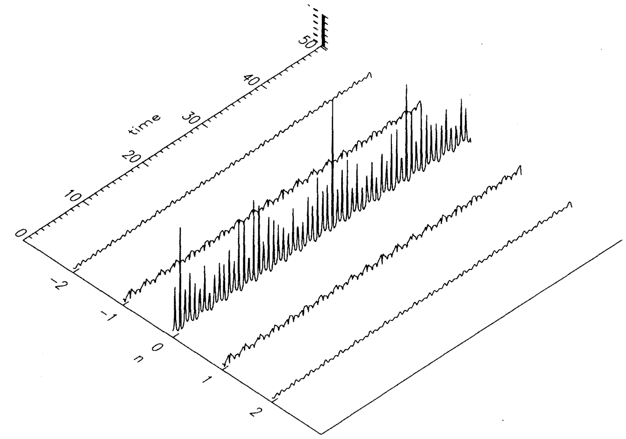

FIG. 1. A discrete breather, Eq. (9), with $\beta = 0.5$. In the plot, each solid line is the time evolution of the modulus $\psi_n(t)$ for the lattice site n.

As a consequence of the Ablowitz-Ladik discretization, the breather solution (9) possesses a continuous translational symmetry, that is, x_0 is an arbitrary real number and its energy and norm are independent of x_0. The energy and norm of the breather are

$$E_b = -\sum_{n=-\infty}^{\infty}(\psi_n^b\psi_{n+1}^{b*} + \psi_n^{b*}\psi_{n+1}^b) = -\frac{4\sinh^2 3\beta}{\sinh\beta}, \quad (15)$$

$$\mathcal{N}_b = \sum_{n=-\infty}^{\infty}\ln(1 + |\psi_n^b|^2) = 18\beta. \quad (16)$$

Since a single soliton, i.e., $\psi_n^s = \sinh\eta\,\mathrm{sech}[\eta(n - x_0)]\exp(-i\omega t)$, where $\omega = -2\cosh\eta$, has the energy and norm

$$E_s(\eta) = -4\sinh\eta, \quad (17)$$

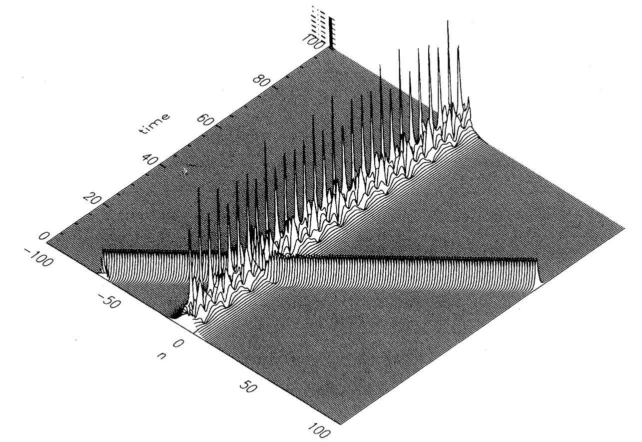

FIG. 2. A discrete breather, Eq. (9), with $\beta=0.3$ collides with a single soliton which has the initial condition, $\sinh\beta' \,\mathrm{sech}[\beta'(n-x_0')]\exp(i\alpha n)$, where $\beta'=1$ and $\alpha=1.5$. The breather remains stable after the collision (see text).

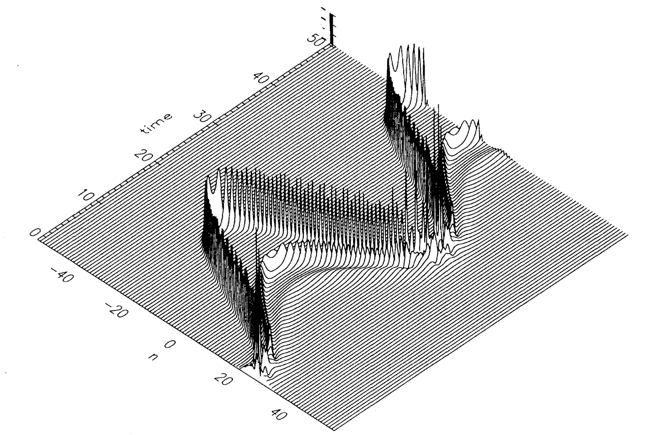

FIG. 3. Time evolution of a discrete breather in the presence of a static linear potential $V_n=\mathscr{E}n$ with $\mathscr{E}=0.2$. The initial wave function, $\sinh(3\beta)\mathrm{sech}[\beta(n-x_0)]$ with $\beta=0.5$, evolves into three spatially separate, coherent structures. Note that the motion is periodic, cf. Fig. 1.

$$\mathscr{N}_s(\eta)=2\eta, \tag{18}$$

respectively, it can readily be verified that the energy and the norm of the breather are the simple arithmetic sum of those of the three single solitons with $\eta=\beta$, 3β, and 5β. These properties indicate that the breather (9) is a special kind of bound state with null binding energy and norm. Therefore it is marginally stable and may decay into the three solitonic components under perturbation. For example, with an initial phase perturbation, i.e., with a phase $\exp(-i\alpha n)$ multiplied with the initial condition (2), as expected, a breather starting with this initial condition will gradually split into three solitons no matter how small α is (see also Ref. [5]). However, it is stable, i.e., remaining as a spatially coherent, single entity if there are only small real-valued perturbations, i.e., amplitude perturbations to Eq. (2) in its initial condition. These qualitative statements can be understood via inverse scattering transformation of the perturbation on initial value problems, and are consistent with our observations in numerical simulations performed with numerical integrations of the full system (1). Moreover, as a more significant property of the solution, the breather (9) is stable after a collision with a soliton in our numerical simulations. Figure 2 shows such a collision event. As usual, there is a space "phase" shift after the collision. However, on account of the smallness of the shift due to the parameter values we chose for the figure, it is rather difficult to perceive for the case presented.

Incidentally, for the system (1) there are staggered breathers. They take their descendency from staggered linear phonons which are situated at the upper frequency edge of the linear phonon band. These breathers evolve from the initial condition, $(-1)^n \sinh(3\beta)\mathrm{sech}[\beta(n-x_0)]$, and have a form which is obtained from Eq. (9) by the transformations, $\psi_n\to(-1)^n\psi_n$ and $t\to -t$ [note that under these transformations, Eq. (1) is invariant].

In the presence of a spatially linear potential, Eq. (1) becomes

$$i\dot{\psi}_n=-(\psi_{n+1}+\psi_{n-1})-\mu(\psi_{n+1}+\psi_{n-1})|\psi_n|^2+V_n\psi_n, \tag{19}$$

where $V_n=\mathscr{E}(t)n$ with $\mathscr{E}(t)$ being any real function of time. This is still a completely integrable system and possesses an infinite number of conservation laws [10]. It can easily be shown that the initial condition (2) also gives rise to a breather solution. However, this breather usually will evolve into spatially separate, coherent structures. For example, as shown in Fig. 3 for a case in which $\mathscr{E}(t)$ is time independent, the breather breaks into three spatially separate, coherent structures, but it refocuses into a single lump periodically as any evolution from a localized initial condition, in this particular type of potential, is periodic [10,11]. As shown in Ref. [8], this breakup is a general discrete lattice effect on the evolution of the breather of this kind in the presence of an external field.

Finally, we point out that the initial condition,

$$\psi_n(t=0)=\sinh(N\beta)\mathrm{sech}[\beta(n-x_0)], \tag{20}$$

should give rise to an N-soliton breather with its poles at $\exp(-k\beta)$, $k=1,3,\ldots,2N-1$, in the IST framework. This breather evolves with $N-1$ internal frequencies in addition to a carrier-wave frequency. It has energy and norm

$$E_{b,N}(\beta)=-\frac{4\sinh^2 N\beta}{\sinh\beta}, \tag{21}$$

$$\mathscr{N}_{b,N}(\beta)=\sum_{n=-\infty}^{\infty}\ln\{1+\sinh^2(N\beta)\mathrm{sech}^2[\beta(n-x_0)]\} =2N^2\beta, \tag{22}$$

respectively. Since the following identities hold,

$$E_{b,N}(\beta)=\sum_{k=1}^{N}E_s[(2k-1)\beta], \tag{23}$$

$$\mathscr{N}_{b,N}(\beta)=\sum_{k=1}^{N}\mathscr{N}_s[(2k-1)\beta], \tag{24}$$

the N-soliton breather is a bound state of marginal stability. This N-soliton breather is obviously a discrete generalization of the continuum N-soliton solution studied in Ref. [5].

To conclude, we have presented an exact solution of a discrete breather which is a spatially localized, temporally quasiperiodic nonlinear coherent excitation, in contrast to the breather commonly studied in the literature, which is a time-periodic excitation. Breathers of multiple internal frequencies are conceptually important in the studies of nonlinear lattice systems. To understand the dynamical properties of the breathers, we have discussed their stability and their behaviors in the presence of an external potential. Finally, we have also indicated how to obtain an N-soliton breather as a discrete generalization of the continuum multiple-soliton solution.

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[1] For example, D.K. Campbell and M. Peyrard, in *Chaos*, edited by D.K. Campbell (American Institute of Physics, New York, 1990), pp. 305–334, and references therein.

[2] A.J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (9188).

[3] For example, S. Takeno, J. Phys. Soc. Jpn. **61**, 2821 (1992), and references therein.

[4] R.S. Mackay and S. Aubry, Nonlinearity **7**, 1623 (1994); see also R.S. Mackay, in *Proceedings of the International Conference on Dynamical Systems and Chaos*, edited by Y. Aizawa, S. Saito, and K. Shiraiwa (World Scientific, New Jersey, 1995), Vol. 2, pp. 34–43.

[5] J. Satsuma and N. Yajima, Suppl. Prog. Theor. Phys. **55**, 284 (1974).

[6] For example, *Future Directions of Nonlinear Dynamics in Physical & Biological Systems*, special issue of Physica **68**D (1993), and also in Vol. 312 of *NATO Advanced Study Institute Series B: Physics*, edited by P.L. Christiansen, J.C. Eilbeck, and R.D. Parmentier (Plenum Press, New York, 1993).

[7] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, New York, 1991).

[8] D. Cai, A.R. Bishop, and N. Grønbech-Jensen, Phys. Rev. E (to be published).

[9] For example, R. Hirota, in *Solitons*, edited by R.K. Bullough and P.J. Caudrey (Springer-Verlag, New York, 1980).

[10] D. Cai, A.R. Bishop, N. Grønbech-Jensen, and M. Salerno, Phys. Rev. Lett. **74**, 1186 (1995).

[11] R. Scharf and A.R. Bishop, Phys. Rev. A **43**, 6535 (1991).